Lecture 2

13 February 2020 17:46

LIE GROUPOIDS

references: Section 2.3.3 of my thesis, Chapter 1 of Mackenzie

Definition Proposition Exercise

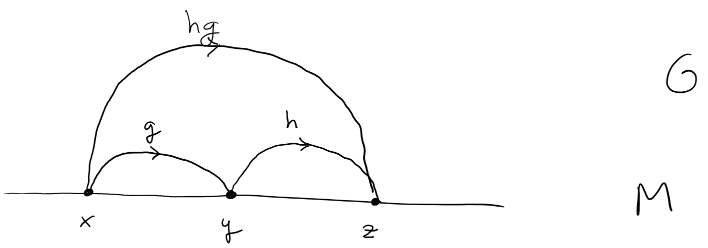
(I) DEFINITION AND STRUCTURE MAPS

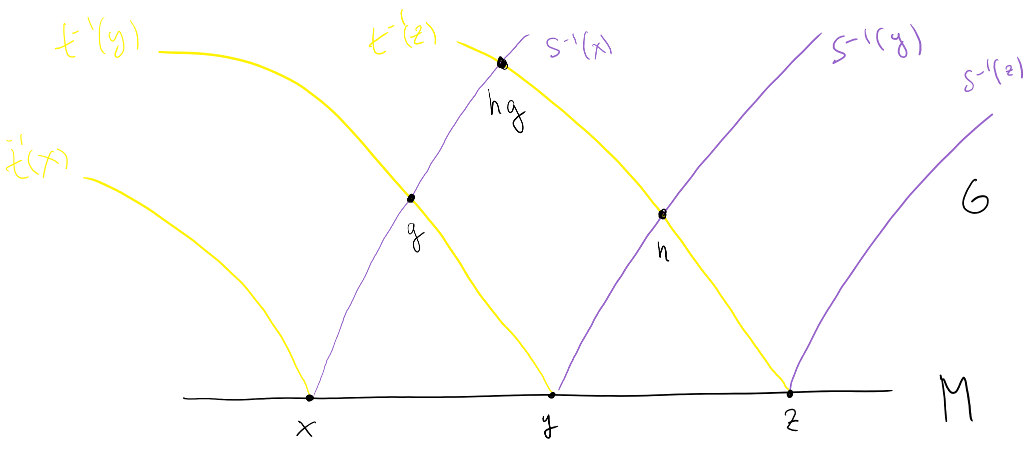
Definition of Groupoid as a (small) category with all invertible morphisms

Set-theoretic content of the definition and structure maps

$$\begin{tikzcd} G \arrow[loop left, "i"] \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] & G \mathbin{{}_s\!\times_t} G \arrow[l, "m"'] \\ M \arrow[u, bend right, "e"'] & \end{tikzcd}$$

Groupoids as sets with partial multiplication: originating from the categorical composition of morphisms





$$g \in G(x,y) = t^{-1}(y) \cap s^{-1}(x)$$
$$h \in G(y,z) = t^{-1}(z) \cap s^{-1}(y)$$
$$hg \in G(x,z) = t^{-1}(z) \cap s^{-1}(x)$$

Right action: $R_g : s^{-1}(y) \to s^{-1}(x)$
$$h \mapsto hg$$

Left action: $L_h : t^{-1}(y) \to t^{-1}(z)$
$$g \mapsto hg$$

Note that these are invertible maps, then we observe that all the s,t-fibres are bijective.

Isotropy $\quad G_x := G(x,x) = t^{-1}(x) \cap s^{-1}(x) \subset G$

Orbit $\quad O_x = \{ t(g),\ g \in s^{-1}(x) \} \subset M$

Orbit set $\quad M/G$

II) Bisections

Bisections $\quad \Gamma(G) = \{ b\ \substack{G \\ \uparrow \\ M}\ \exists\, \sigma_b, \tau_b \mid s \circ b = \sigma_b,\ t \circ b = \tau_b,\ \tau_b^{-1} \sigma_b \tau_b = id_M \}$

Proposition 2.1 Bisections form a group

The group operation is given point-wise by:

$$a, b \in \Gamma(G), \quad (a \cdot b)(x) = a(\tau_b(x))\, b(\tau_b^{-1} \circ \sigma_a \circ \tau_b(x))$$

Proof.

1. well-definedness: $gh \in G$ requires $s(g) = t(g)$

$$\checkmark \left(\begin{array}{l} s(a(\tau_b(x))) = \sigma_a \tau_b(x) \\ t(b(\tau_b^{-1}\sigma_a \tau_b(x))) = \tau_b \tau_b^{-1} \sigma_a \tau_b(x) = \sigma_a \tau_b(x) \end{array} \right.$$

2. associativity

given by associativity of the groupoid multiplication

3. identity

$e\ \substack{G \\ \uparrow \\ M}$ identity map $\quad s \circ e = id_M = t \circ e$

$$e \cdot a\ (x) = e(\tau_a(x))\, a(\tau_b^{-1} id_M \tau_b(x)) = e(\tau_b(x))\, a(x) = a(x)$$

$$a \cdot e\ (x) = a(id_M(x))\, e(id_M \sigma_a\, id_M(x)) = a(x) e(\sigma_a(x)) = a(x)$$

4. inverse

$$a^{-1}(x) = a(\tau_a^{-1}\sigma_a(x))^{-1}$$

$$\begin{aligned} a^{-1} \cdot a\ (x) &= a^{-1}(\tau_a(x))\, a(\tau_a^{-1}\sigma_a \tau_a(x)) \\ &= (a(\tau_a^{-1}\sigma_a \tau_a(x)))^{-1} a(\tau_a^{-1}\sigma_a \tau_a(x)) \\ &= e(\tau_a^{-1}\sigma_a \tau_a(x)) = e(x) \end{aligned}$$

Left (and right) translations

Given a bisection we can define the equivalent of left and right actions:

$$b \in \Gamma(G) \qquad L_b : G \to G$$

$$g \mapsto b(t(g))\, g$$

III Lie Groupoids

Lie groupoid : demand that structure sets and maps are smooth together with the necessary requirement that $s,t : G \to M$ are submersions

A few automatic consequences: i, L_g, R_g become diffeomorphisms, a embedding

$G_x \subset G$ are Lie groups, $O_x \underset{imm}{\subset} M$, $t : s^{-1}(x) \to O_x$ / $s : t^{-1}(x) \to O_x$ principal G_x-bundle

Smooth (local) bisections $\Gamma(G)$ with σ, τ diffeomorphisms

Proposition 2.2 $L_- : \Gamma(G) \to \mathrm{Diff}(G)$ is a group morphism

Proof.

IV Examples of Lie Groupoids

- Lie groups $\begin{matrix} G \\ \downarrow\downarrow \\ \{x\} \end{matrix}$ $\quad G \cong G_x \cong \Gamma(G)$ as groups
- Manifolds : pair groupoid $\begin{matrix} M \times M \\ pr_1 \downarrow\downarrow pr_2 \\ M \end{matrix}$ — Smooth equivalence relations $R \subset M \times M$
- Path/Fundamental groupoid $\Pi(M) = \{ [\gamma]_{homotopy}, \gamma : [0,1] \to M \}$ $\quad \Pi_x(M) \cong \pi_1(M,x)$, $\Gamma(\Pi(M)) \cong \mathrm{Diff}(M)$
- General Linear groupoid $GL(E) = \{ \psi_{xy} : E_x \cong E_y \,,\ x,y \in M \}$ $\quad GL_x(E) \cong GL(E_x)$, $\Gamma(\Pi(M)) \cong \mathrm{Aut}(E)$

V Morphisms of Lie Groupoids

Morphism of Lie groupoids as smooth functors

$$\begin{matrix} \Phi : & G & \longrightarrow & H \\ & \downarrow\downarrow & & \downarrow\downarrow \\ \varphi : & M & \longrightarrow & N \end{matrix}$$

$$\Phi(gh) = \Phi(g)\Phi(h)$$
$$\Phi \circ e_G = e_H \circ \varphi$$
$$\Phi \circ i_G = i_H \circ \Phi$$

Proposition 2.3 Lie groupoid morphisms covering diffeo. induce group morphisms of bisections

Proof.

VI Vector Fields on Lie Groupoids

s-Vertical vector field

Left-Invariant vector fields

Proposition 2.4 s-vertical LI vector fields are involutive